Tutorial 11

April 20, 2017

1. Prove Dirichlet's principle for the Neumann boundary condition. It asserts that among *all* real-valued functions $w(\mathbf{x})$ on D the quantity

$$E[w] = \frac{1}{2} \iiint\limits_{D} |\nabla w|^2 d\mathbf{x} - \iint\limits_{\text{bdy } D} hw \, dS$$

is the smallest for $w = u$, where u is the solution of the Neumann problem

$$-\Delta u = 0 \text{ in } D, \qquad \frac{\partial u}{\partial n} = h(\mathbf{x}) \text{ on bdy } D.$$

It is required to assume that the average of the given function $h(\mathbf{x})$ is zero (by Exercise 6.1.11).

Notice three features of this principle:

(i) There is *no constraint at all* on the trial functions $w(\mathbf{x})$.

(ii) The function $h(\mathbf{x})$ appears in the energy.

(iii) The functional $E[w]$ does not change if a constant is added to $w(\mathbf{x})$.

(*Hint:* Follow the method in Section 7.1.)

Solution: Suppose $u(\mathbf{x})$ solves the above problem, w is any function and let $v = u - w$, then

$$\begin{aligned}
E[w] = E[u-v] &= E[u] - \iiint_D \nabla u \cdot \nabla v d\mathbf{x} + \iint_{\partial D} hv dS + \frac{1}{2} \iiint_D |\nabla v|^2 d\mathbf{x} \\
&= E[u] - \iint_{\partial D} \frac{\partial u}{\partial n} v dS + \iiint_D \Delta u v d\mathbf{x} + \iint_{\partial D} hv dS + \frac{1}{2} \iiint_D |\nabla v|^2 d\mathbf{x} \\
&= E[u] + \frac{1}{2} \iiint_D |\nabla v|^2 d\mathbf{x}
\end{aligned}$$

which implies

$$E[w] \geq E[u].$$

2. Give yet another derivation of the mean value property in three-dimensions by choosing D to be a ball and x_0 its center in the representation formula (1).

Solution: Choosing $D = B(\mathbf{x}_0, R)$ in the representation formula (1) and using the divergence theorem,

$$\begin{aligned}
u(\mathbf{x}_0) &= \iint\limits_{\partial B(\mathbf{x}_0, R)} \left[-u(\mathbf{x}) \cdot \frac{\partial}{\partial n} \left(\frac{1}{|\mathbf{x} - \mathbf{x}_0|} \right) + \frac{1}{|\mathbf{x} - \mathbf{x_0}|} \cdot \frac{\partial u}{\partial n} \right] \frac{dS}{4\pi} \\
&= \iint\limits_{|\mathbf{x} - \mathbf{x}_0| = R} \left[\frac{1}{R^2} u(\mathbf{x}) + \frac{1}{R} \frac{\partial u}{\partial n} \right] \frac{dS}{4\pi} \\
&= \frac{1}{4\pi R^2} \iint\limits_{|\mathbf{x} - \mathbf{x}_0| = R} u dS + \frac{1}{4\pi R} \iiint\limits_{|\mathbf{x} - \mathbf{x}_0| < R} \Delta u d\mathbf{x} \\
&= \frac{1}{4\pi R^2} \iint\limits_{|\mathbf{x} - \mathbf{x}_0| = R} u dS.
\end{aligned}$$

3. **Theorem 2 on P181**: The solution of the problem

$$\Delta u = f \quad \text{in } D \quad u = h \ \text{ on } \partial D$$

is given by

$$u(\mathbf{x}_0) = \iint_{\partial D} h(\mathbf{x}) \frac{\partial G(\mathbf{x}, \mathbf{x}_0)}{\partial n} dS + \iiint_D f(\mathbf{x}) G(\mathbf{x}, \mathbf{x}_0) d\mathbf{x}$$

Solution: Let $v(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0|}, \mathbf{x} \neq \mathbf{x}_0$, then $\Delta v(\mathbf{x}) = 0, \mathbf{x} \neq \mathbf{x}_0$. Let $D_\epsilon = D \setminus B_\epsilon(x_0)$. Applying Green's Second Identity to v and u on D_ϵ, we have

$$\iiint_{D_\epsilon} -vf d\mathbf{x} = \iiint_{D_\epsilon} u\Delta v - v\Delta u d\mathbf{x} = \iint_{\partial D_\epsilon} \left[u \cdot \frac{\partial v}{\partial n} - \frac{\partial u}{\partial n} \cdot v \right] dS$$

Noting that ∂D_ϵ consists of two parts and on $\{|\mathbf{x} - \mathbf{x}_0| = r = \epsilon\}$, $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$, we have

$$\iint_{r=\epsilon} u\frac{\partial v}{\partial n} - \frac{\partial u}{\partial n} v dS = -\iint_{r=\epsilon} u\frac{\partial v}{\partial r} - \frac{\partial u}{\partial r} v dS = -\frac{1}{4\pi\epsilon^2} \iint_{r=\epsilon} u dS - \frac{1}{4\pi\epsilon} \iint_{r=\epsilon} \frac{\partial u}{\partial r} dS = -\bar{u} - \epsilon \overline{\frac{\partial u}{\partial r}}$$

where $\overline{u}$ denotes the average value of u on the sphere $\{r = c\}$, and $\overline{\frac{\partial u}{\partial r}}$ denotes the average value of $\frac{\partial u}{\partial r}$ on this sphere. Since u is continuous and $\frac{\partial u}{\partial r}$ is bounded, we have

$$-\overline{u} - \epsilon \overline{\frac{\partial u}{\partial r}} \to -u(\mathbf{x}_0) \quad \text{as } \epsilon \to 0.$$

So let ϵ tend to 0 and then we have

$$\iiint_D -vf d\mathbf{x} = \iint_{\partial D} \left[u \cdot \frac{\partial v}{\partial n} - \frac{\partial u}{\partial n} \cdot v \right] dS - u(\mathbf{x}_0) \tag{1}$$

Suppose $G(\mathbf{x}, \mathbf{x}_0)$ is the Green's function for $-\Delta$, then $H = G - v$ is a harmonic function on D, and $G = 0$ on ∂D. Applying the second Green's Identity to u and H on D, we have

$$\iiint_D -Hf d\mathbf{x} = \iiint_D u\Delta H - H\Delta u d\mathbf{x} = \iint_{\partial D} \left[u \cdot \frac{\partial H}{\partial n} - \frac{\partial u}{\partial n} \cdot H \right] dS \tag{2}$$

Adding (2) and (3) and using $G = H + v$ in D,$G = 0$ on ∂D , we get

$$\iiint_D -Gf d\mathbf{x} = \iint_{\partial D} \left[u \cdot \frac{\partial G}{\partial n} - \frac{\partial u}{\partial n} \cdot G \right] dS - u(\mathbf{x}_0) = \iint_{\partial D} h \frac{\partial G}{\partial n} dS - u(\mathbf{x}_0)$$

That is,

$$u(\mathbf{x}_0) = \iint_{\partial D} h \frac{\partial G}{\partial n} dS + \iiint_D Gf d\mathbf{x}$$